Transition Temperature in QCD
with physical light and strange quark masses
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Introduction
Thermodynamics on lines of constant physics, improved actions

The transition temperature on $N_\tau = 4, 6$ lattices
extrapolation to physical quark masses and continuum extrapolation

Equation of state on $N_\tau = 4, 6$ lattices
quark mass and cut-off dependence, crossover-region

Conclusions

†based on:
2) RBC-Bielefeld collaboration, in preparation
**Introduction**

**Goal:** QCD thermodynamics with realistic quark masses and controlled extrapolation to the continuum limit \( T_c, \text{EoS, } \mu_q > 0, \ldots \)

- use an improved staggered fermion action that removes \( \mathcal{O}(a^2) \) errors in bulk thermodynamic quantities and reduces flavor symmetry breaking inherent to the staggered formulation

  - **RBC-Bielefeld choice:** p4-action + 3-link smearing (p4fat3)
  - **MILC:** Naik-action + (3,5,7)-link smearing (asqtad);
  - **Wuppertal:** standard staggered + exponentiated 3-link smearing (stout)
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RBC-Bielefeld choice: p4-action + 3-link smearing (p4fat3)

- p4-action: smooth high-T behavior for bulk thermodynamics on lattice with temporal extent $N_\tau$

$$p(N_\tau)/T^4 = p_{SB}^{cont}/T^4 + \mathcal{O}(N^{-4}_\tau)$$

- p4&Naik: small cut-off dependence of renormalized Polyakov loops (see K. Petrov) and quark number susceptibilities
Introduction

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- use an improved staggered fermion action that removes $\mathcal{O}(a^2)$ errors in bulk thermodynamic quantities and reduces flavor symmetry breaking inherent to the staggered formulation
- RBC-Bielefeld choice: p4-action + 3-link smearing (p4fat3)
- use the newly developed RHMC algorithm to remove ‘step-size errors’ in the numerical simulation
- perform simulations with (3-4) different light quark masses corresponding to $150 \text{ MeV} \lesssim m_\pi \lesssim 500 \text{ MeV}$ at 2 different values of the lattice cut-off controlled by the spatial lattice size $N_\tau = 4, 6$ to perform the chiral and continuum extrapolation

previous results with p4-action:
- 2-flavor QCD: $N_\tau = 4$, $m_\pi \simeq 770 \text{ MeV}$
Thermodynamics on QCDOC and apeNEXT

US/RBRC QCDOC
20,000,000,000,000 ops/sec

BI – apeNEXT
5,000,000,000,000 ops/sec

- critical temperature
- equation of state
- finite density QCD
Transition temperature

- simulations at several temperatures ($\beta$-values) in a **narrow temperature range** to insure the applicability of Ferrenberg-Swendsen reweighting; combines information from several independent simulations

- determine location of the transition from various susceptibilities: (disconnected part of the) light and strange quark chiral susceptibility; Polyakov loop and quark number susceptibility,... requires **large statistics; several ten thousand trajectories**

- differences in location of transition are small

- runs on different size volumes; **confirm small finite volume effects**; find little influence on location of transition

- overall scale setting using $T = 0$ potential parameter; **find weak cut-off dependence**, well understood in the pure gauge theory

- verify that $T_c$ determination is consistent with crossover region in the energy density (ongoing)
Chiral susceptibility, $\mathcal{N}_\tau = 4, 6$

![Graphs showing chiral susceptibility](image)

- weak volume dependence
- peak location consistent with that of Polyakov loop susceptibility and maximum of quartic fluctuation of quark number density
Chiral susceptibility, $\mathcal{N}_\tau = 4$
Chiral susceptibility, $\mathcal{N}_\tau = 4$

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2.5% error band $\Leftrightarrow$ 5 MeV

Data sample for smallest quark mass on $16^3 \times 4$ lattice

200,000/10 trajectories enter Ferrenberg-Swendsen sample
Chiral susceptibility, $\mathcal{N}_\tau = 6$

Data sample for smallest quark mass on $16^3 \times 6$ lattice

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4% error band $\Leftrightarrow$ 8 MeV

260,000/10 trajectories enter Ferrenberg-Swendsen sample
Ambiguities in locating the crossover point

Differences of pseudo-critical couplings locating peaks in light ($\beta_l$), strange ($\beta_s$) and Polyakov loop ($\beta_L$) susceptibilities

$2.5\% \left( N_\tau = 4 \right) \text{ or } 4\% \left( N_\tau = 6 \right)$

error band $\Leftrightarrow 5 \text{ or } 8 \text{ MeV}$

differences in the location of pseudo-critical couplings are taken into account as systematic error
Strangeness fluctuations provide yet another completely different observable to locate the crossover point.

Quartic strangeness fluctuations are strongly peaked in the transition region.

\[
d_4^s = \frac{1}{V T^3} \frac{\partial^4 \ln Z}{\partial (\mu_s/T)^4}
\]

\(T_c\) determined from peak in chiral susceptibility.

Differences of pseudo-critical couplings deduced from peak in strangeness and Polyakov loop fluctuations:

\[
\beta_c(\chi_L) - \beta_c(\chi_s) = 0.001(2)
\]

Similar for light quark number fluctuations:

\[
\beta_c(\chi_L) - \beta_c(\chi_{u,d}) = 0.005(3)
\]
$T = 0$ scale setting using the heavy quark potential

use $r_0$ or string tension to set the scale for $T_c = 1/N_\tau a(\beta_c)$

$$V(r) = -\frac{\alpha}{r} + \sigma r , \quad r^2 \frac{dV(r)}{dr}|_{r=r_0} = 1.65$$

no significant cut-off dependence when cut-off varies by a factor 4

i.e. from the transition region on $N_\tau = 4$ lattices to that on $N_\tau = 16$ lattices !!

we use $r_0 = 0.469(7)$ fm determined from quarkonium spectroscopy

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$r_0 \sqrt{\sigma}$ fluctuates by about 3% in this interval

no hint for large $O(a^2)$ corrections

we use $r_0 = 0.469(7)$ fm determined from quarkonium spectroscopy

\( \Rightarrow T_c r_0, \ T_c / \sqrt{\sigma} \)

extrapolation to chiral and continuum limit

\[
(r_0 T_c)_{N_f} = (r_0 T_c)_{\text{cont.}} + b (m_{PS} r_0)^d + c / N_f^2
\]

\(d=1.08\ (O(4),\ 2\text{nd ord.}),\ d=2\ (1\text{st ord.})\)

\(\Rightarrow r_0 T_c = 0.456(7)^{+3}_{-1},\ \ T_c / \sqrt{\sigma} = 0.408(7)^{+3}_{-1}\) at phys. point

\(\Rightarrow T_c = 192(7)^{\pm 4}\) MeV

(1st error: stat. error on \(\beta_c\) and \(r_0\); 2nd error: \(N_f^{-2}\) extrapolation)
Transition temperature: $\mathcal{N}_\tau = 4, 6$

- RBC-Bielefeld (p4fat3 (p4)) vs. MILC (asqtad (Naik)) and Wuppertal (stout (stand. staggered))

- asqtad results for $\mathcal{N}_\tau = 4$ and 6 agree with p4 results within statistical errors; 
  (C. Bernard et al., PR D71, 034504 (2005))

- results obtained with stout action for $\mathcal{N}_\tau = 4$ and 6 are about 15% lower; 
  $\beta_c$ from $\mathcal{N}_\tau = 8, 10$ covers $(151 - 176)$ MeV; 
  (Y. Aoki et al., hep-lat/0609068)

\[
\begin{align*}
\text{MILC data for } T_c r_1 & \text{ rescaled with } r_0/r_1 = 1.4795
\end{align*}
\]
extrapolations to phys. point

- RBC-Bielefeld (p4fat3 (p4)) vs. Wuppertal (stout (stand. staggered))
- results for $N_\tau = 4, 6$ differ by 15% but show similar cut-off dependence
- stout results for different observables no longer consistent with each other for $N_\tau = 8, 10$

overall scale set with $r_0 = 0.469 \text{ fm}$
Calculating the EoS on lines of constant physics (LCP)

The pressure

\[
\frac{p}{T^4} \bigg|_{\beta_0}^{\beta} = N_f^4 \int_{\beta_0}^{\beta} d\beta' \left[ \frac{1}{N_f^3 N_t} (\langle S_g \rangle_0 - \langle S_g \rangle_T) 
- \left( 2(\langle \bar{\psi}\psi \rangle_{l0} - \langle \bar{\psi}\psi \rangle_{lT}) + \frac{\hat{m}_s}{\hat{m}_l} (\langle \bar{\psi}\psi \rangle_{s0} - \langle \bar{\psi}\psi \rangle_{sT}) \right) \frac{\partial \hat{m}_l}{\partial \beta'} \hat{m}_s / \hat{m}_l 
- \hat{m}_l (\langle \bar{\psi}\psi \rangle_{s0} - \langle \bar{\psi}\psi \rangle_{sT}) \frac{\partial (\hat{m}_s / \hat{m}_l)}{\partial \beta'} \hat{m}_l \right]
\]

The interaction measure for \( N_f = 2 + 1 \)

\[
\frac{\epsilon - 3p}{T^4} = T \frac{d}{dT} \left( \frac{p}{T^4} \right) = \left( a \frac{d\beta}{da} \right)_{LCP} \frac{\partial p / T^4}{\partial \beta} = \left( \frac{\epsilon - 3p}{T^4} \right)_{gluon} + \left( \frac{\epsilon - 3p}{T^4} \right)_{fermion} + \left( \frac{\epsilon - 3p}{T^4} \right) \hat{m}_s / \hat{m}_l
\]
a precise knowledge of $r_0/a$ is mandatory to fix the temperature scale and the $\beta$-function, $(a d \beta / d a)_{LCP}$

RG-inspired ansatz for $\beta > 3.47$; rational polynomial ansatz for $\beta < 3.47$

particularly important in the transition region on coarse lattices such as $N_\tau = 4, 6$

**NOTE:**

$r_0 T = (r_0/a)/N_\tau$ defines the temperature scale irrespective of any ambiguities in locating $T_c$.  

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RBC-Bielefeld, preliminary
$(\epsilon - 3p)/T^4$ on LCP

Using an RG-inspired 2-loop $\beta$-function underestimates $(\epsilon - 3p)/T^4$ in the transition region and stretches the temperature interval in the low temperature regime artificially, i.e. makes the transition region look broader than it is.

Differences in the transition region partly arise from differences in the $\beta$-functions used in the crossover region.

- Overall good agreement

Note:
$T$-scale is not dependent on $T_c$ determination

RBC-Bielefeld, preliminary

asqtad data:
C. Bernard et al., hep-lat/0610017 (Lattice'06)
Energy density and pressure

$N_\tau = 4, 6$

**RBC-Bielefeld vs. MILC:** the RBC-Bi energy density on $N_\tau = 4$ lattices rises more steeply; direct consequence of the use of a non-perturbative $\beta$-function directly deduced from calculated $r_0/a$ values

**overall good agreement for $N_\tau = 4, 6$,**

*Note:* $T$-scale does not depend on $T_c$ determination!!

- Band marks $T = (192 \pm 11)$ MeV
- RBC-Bielefeld, preliminary

Pressure increased slightly with smaller quark mass
Conclusions

- We studied the thermodynamics of QCD with a realistic quark mass spectrum and performed extrapolations to the physical point using an improved staggered fermion action (p4fat3) we find from an analysis at 2 different lattice spacings and for several values of the quark mass consistent crossover temperatures from several observables.

- At the physical point of (2+1)-flavor QCD we obtain the continuum extrapolated value:
  \[ T_c = 192(7)(4)\text{MeV} \]

- the analysis of \( \epsilon/T^4 \) shows a rapid crossover at this temperature;

- this analysis of bulk thermodynamics is consistent with an analysis based on the asqtad action performed by MILC details in the transition region still have to be resolved ⇒ hotQCD-collaboration (see poster by R. Soltz)