

Chromohydrodynamics of the Quark-Gluon Plasma

Stanisław Mrówczyński

*Świętokrzyska Academy, Kielce, Poland
& Institute for Nuclear Studies, Warsaw, Poland*

in collaboration with Cristina Manuel

C. Manuel & St. M., Phys. Rev. **D74** (2006) 105003

Motivation

Fluid equations are frequently applied to the electromagnetic plasma

Why not to the quark-gluon plasma?

**Long but quiet history
of chromohydrodynamics**

Kajantie & Montonen 1980
Holm & Kupersmidt 1984
Mrówczyński 1988
Bhatt, Kaw & Parikh 1989
Jackiw, Nair & Pi 2000
Manuel & Mrówczyński 2003

Real hydrodynamics of QGP

Real hydrodynamics describes systems in local thermodynamic equilibrium

- Local equilibrium is a state of maximal local entropy $C=0$
- Local equilibrium is achieved at the time scale of hard collisions $\sim (g^4 \ln(1/g) T)^{-1}$
- Color redistribution occurs at the time scale of hard collisions $\sim (g^2 \ln(1/g) T)^{-1}$
- Whitening occurs at the time scale of soft collisions $\sim \ln(1/g) / T$

Manuel & Mrówczyński,
Local Equilibrium of the Quark-Gluon Plasma,
Phys. Rev. **D68** (2003) 094010

Manuel & Mrówczyński,
Whitening of the Quark-Gluon Plasma,
Phys. Rev. **D70** (2004) 094019

Real hydrodynamics of QGP cont.

Real hydrodynamics of QGP is colorless

$$t_{\text{hydro}} \geq t_{\text{hard coll}} \sim \left(g^4 \ln(1/g) T \right)^{-1}$$

There is no QGP analog of magnetohydrodynamics of EM plasma

Delayed mutual equilibration of electrons and ions

$$m_{\text{ion}} \gg m_{\text{electron}}$$

Chromohydrodynamics

Fluid approach to QGP at time scale shorter than soft collisions

$$t < t_{\text{soft coll}} \sim \left(g^2 \ln(1/g) T \right)^{-1}$$

Collisionless regime of kinetic theory

Kinetic theory

Distribution functions of quarks $Q(p, x)$ and antiquarks $\bar{Q}(p, x)$ are $N_c \times N_c$ matrices

Distribution function of gluons $G(p, x)$ is $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix

Transport equations

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q\} = C$$

quarks

$$p_\mu D^\mu \bar{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu \bar{Q}\} = \bar{C}$$

antiquarks

adjoint

$$p_\mu D^\mu G - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu G\} = C_g$$

gluons

$$D^\mu \equiv \partial^\mu - ig[A^\mu, \dots]$$

free streaming

mean-field force

collisions

Mean-field generation

$$D_\mu F^{\mu\nu} = j^\nu[Q, \bar{Q}, G]$$

$$D^\mu \equiv \partial^\mu F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

Collisionless limit: $C = \bar{C} = C_g = 0$

Streams

$$Q(p, x) = \sum_{\alpha} Q_{\alpha}(p, x)$$

$$p_{\mu} D^{\mu} Q_{\alpha} - \frac{g}{2} p^{\mu} \{F_{\mu\nu}, \partial_p^{\nu} Q_{\alpha}\} = 0$$

Streams 'talk' to each other only via mean field.

$$D_{\mu} F^{\mu\nu} = \sum_{\alpha} j_{\alpha}^{\nu}$$

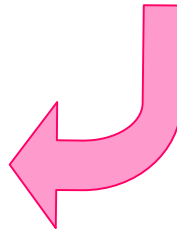
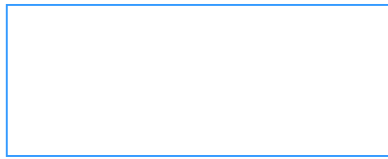
Streams

Derivation of chromohydrodynamic equations

Transport equation of quark distribution function of the α -th stream $Q_\alpha(p, x)$

$$p_\mu D^\mu Q_\alpha(p, x) - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q_\alpha(p, x)\} = 0$$

$\int dP$



Taking into account antiquarks and gluons is straightforward

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0) \delta(p^2)$$

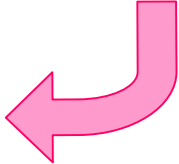


$$D_\mu n_\alpha^\mu(x) = 0$$

$$n_\alpha^\mu(x) \equiv \int dP p^\mu Q_\alpha(p, x)$$

Derivation of chromohydrodynamic equations cont.

$$p_\mu D^\mu Q_\alpha(p, x) - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q_\alpha(p, x)\} = 0$$

$$\int dP p^\mu \boxed{}$$




$$D_\mu T_\alpha^{\mu\nu}(x) - \frac{g}{2} \{F_{\mu\nu}, n_\alpha^\mu(x)\} = 0$$

$$T_\alpha^{\mu\nu}(x) \equiv \int dP p^\mu p^\nu Q_\alpha(p, x)$$

$$m = 0 \Rightarrow T_{\alpha\mu}^\mu = 0$$

Chromohydrodynamic equations

$$D_{\mu} n_{\alpha}^{\mu}(x) = 0$$

$$D_{\mu} T_{\alpha}^{\mu\nu}(x) - \frac{g}{2} \{F^{\mu\nu}, n_{\alpha\mu}(x)\} = 0$$

Postulated form of $n_{\alpha}^{\mu}(x)$ and $T_{\alpha}^{\mu\nu}(x)$: isotropy in the local rest frame

$$n_{\alpha}^{\mu}(x) = n_{\alpha}(x) u_{\alpha}^{\mu}(x)$$

$$T_{\alpha}^{\mu\nu}(x) = \frac{1}{2} (\varepsilon_{\alpha}(x) + p_{\alpha}(x)) \{u_{\alpha}^{\mu}(x), u_{\alpha}^{\nu}(x)\} - p_{\alpha}(x) g^{\mu\nu}$$

$n_{\alpha}(x)$, $\varepsilon_{\alpha}(x)$, $p_{\alpha}(x)$, $u_{\alpha}^{\mu}(x)$ matrices!

Chromohydrodynamic equations cont.

Gauge transformations of $n_\alpha(x)$, $\varepsilon_\alpha(x)$, $p_\alpha(x)$, $u_\alpha^\mu(x)$

$$n_\alpha(x) \rightarrow U(x)n_\alpha(x)U^{-1}(x)$$

$n_\alpha(x)$, $\varepsilon_\alpha(x)$, $p_\alpha(x)$, $u_\alpha^\mu(x)$ cannot be simultaneously diagonalized

$$n_\alpha^\mu(x) \equiv \int dP p^\mu Q_\alpha(p, x)$$

$$Q_\alpha(p, x) \rightarrow U(x)Q_\alpha(p, x)U^{-1}(x)$$

Chromohydrodynamic equations cont.

index α suppressed

$$D_{\mu} n^{\mu}(x) = 0$$

1 matrix equation

$$D_{\mu} T^{\mu\nu}(x) - \frac{g}{2} \{F^{\mu\nu}, n_{\mu}(x)\} = 0$$

4 matrix equations

5 matrix equations

$$n^{\mu}(x) = n(x) u^{\mu}(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2} (\varepsilon(x) + p(x)) \{u^{\mu}(x), u^{\nu}(x)\} - p(x) g^{\mu\nu}$$

$n(x), \varepsilon(x), p(x), u^{\mu}(x)$ 6 matrix functions

$$u^{\mu}(x) u_{\mu}(x) = 1$$

To close the system of equations:

$$\nabla p = 0 \quad \text{or} \quad \varepsilon = 3p \iff T_{\mu}^{\mu} = 0$$

Linear response approximation

Small perturbation of the space-time homogeneous & colorless state

$$n(x) = \tilde{n} + \delta n(x), \quad \varepsilon(x) = \tilde{\varepsilon} + \delta\varepsilon(x),$$
$$p(x) = \tilde{p} + \delta p(x), \quad u^\mu(x) = \tilde{u}^\mu + \delta u^\mu(x)$$

$\tilde{n}, \tilde{\varepsilon}, \tilde{p}, \tilde{u}^\mu$ unit matrices in color space

$$\tilde{n} \gg \delta n, \quad \tilde{\varepsilon} \gg \delta\varepsilon, \quad \tilde{p} \gg \delta p, \quad \tilde{u}^\mu \gg \delta u^\mu$$

$$F^{\mu\nu} \sim A^\mu \sim \delta n$$

Linearized chromohydrodynamic equations

Continuity equation

$$(D_\mu \delta n) \tilde{u}^\mu + \tilde{n} D_\mu \delta u^\mu = 0$$

Euler equation

$$(\tilde{\varepsilon} + \tilde{p}) \tilde{u}^\mu D_\mu \delta u^\nu - (D^\nu - \tilde{u}^\nu \tilde{u}^\mu D_\mu) \delta p - g \tilde{n} \tilde{u}_\mu F^{\mu\nu} = 0$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - \tilde{u}^\mu \tilde{u}^\nu$$

$\delta\varepsilon$ exactly eliminated

$D^\nu \delta p \approx 0$ dynamics dominated by the mean field

Solutions of the linearized equations

- $D^\mu \rightarrow \partial^\mu$ full linearization $A^\mu \sim \delta n$
- Fourier transformations

continuity

$$k_\mu \tilde{u}^\mu \delta n(k) + \tilde{n} k_\mu \delta u^\mu(k) = 0$$

Euler

$$i(\tilde{\varepsilon} + \tilde{p}) \tilde{u}^\mu k_\mu \delta u^\nu(k) - g \tilde{n} \tilde{u}_\mu F^{\mu\nu}(k) = 0$$

$$\partial^\nu \delta p \approx 0$$

Solutions

$$\delta n(k) = ig \frac{\tilde{n}^2}{\tilde{\varepsilon} + \tilde{p}} \frac{\tilde{u}_\nu k_\mu}{(\tilde{u} \cdot k)^2} F^{\mu\nu}(k)$$

$$\delta u^\mu(k) = ig \frac{\tilde{n}}{\tilde{\varepsilon} + \tilde{p}} \frac{\tilde{u}_\nu}{\tilde{u} \cdot k} F^{\mu\nu}(k)$$

index α suppressed

Color current

$$j^\mu(x) = -\frac{g}{2} \left(n(x) u^\mu(x) - \frac{1}{N_c} \text{Tr}[n(x) u^\mu(x)] \right)$$

$$j^\mu(x) = \tilde{j}^\mu + \delta j^\mu(x), \quad \tilde{j}^\mu = 0$$

$$\delta j^\mu(x) = -\frac{g}{2} (\tilde{n} \delta u^\mu(x) + \tilde{u}^\mu \delta n(x))$$

$$\text{Tr}[F^{\mu\nu}] = 0$$

polarization tensor

$$\Pi^{\mu\nu}(x, y) = \frac{\delta j^\mu(x)}{\delta A_\nu(y)}$$

Polarization tensor

index α restored

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \sum_{\alpha} \frac{\tilde{n}_{\alpha}^2}{\tilde{\epsilon}_{\alpha} + \tilde{p}_{\alpha}} \frac{(\tilde{u}_{\alpha} \cdot k)(\tilde{u}_{\alpha}^{\mu} k^{\nu} + \tilde{u}_{\alpha}^{\nu} k^{\mu}) - k^2 \tilde{u}_{\alpha}^{\mu} \tilde{u}_{\alpha}^{\nu} - (\tilde{u}_{\alpha} \cdot k)^2 g^{\mu\nu}}{(\tilde{u}_{\alpha} \cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_{\mu} \Pi^{\mu\nu}(k) = 0$$

Hydrodynamics vs. kinetic theory

Kinetic theory result

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int dP f(p) \frac{(p \cdot k)(p^\mu k^\nu + p^\nu k^\mu) - k^2 p^\mu p^\nu - (p \cdot k)^2 g^{\mu\nu}}{(p \cdot k)^2}$$



$$f(p) = \sum_{\alpha} \tilde{n}_{\alpha} \tilde{u}_{\alpha}^0 \delta^{(3)}(\mathbf{p} - m_{\alpha} \tilde{\mathbf{u}}_{\alpha})$$

$$m_{\alpha} \equiv \frac{\tilde{\varepsilon}_{\alpha} + \tilde{p}_{\alpha}}{\tilde{n}_{\alpha}}$$



Hydrodynamic result

Effect of pressure gradients

The set of fluid equations is closed by the relation $p_\alpha(x) = \frac{1}{3} \varepsilon_\alpha(x)$

$$\Pi^{\mu\nu}(k) = -\frac{3g^2}{8} \sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha} \left[\frac{(\tilde{u}_\alpha \cdot k)(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - k^2 \tilde{u}_\alpha^\mu \tilde{u}_\alpha^\nu - (\tilde{u}_\alpha \cdot k)^2 g^{\mu\nu}}{(\tilde{u}_\alpha \cdot k)^2} \right. \\ \left. - \frac{(\tilde{u}_\alpha \cdot k)k^2(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - (\tilde{u}_\alpha \cdot k)^2 k^\mu k^\nu - k^4 \tilde{u}_\alpha^\mu \tilde{u}_\alpha^\nu}{k^2 + 2(\tilde{u}_\alpha \cdot k)^2} \right]$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Application

Collective modes in the two-stream systems

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] A_\nu(k) = 0$$

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

$$k_\mu \Pi^{\mu\nu}(k) = 0 \quad \downarrow \quad \varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2} \Pi^{ij}(k) \quad \text{chromodielectric tensor}$$

$$\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\omega_p^2 \equiv \frac{g^2}{2} \sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha}$$

$$\varepsilon^{ij}(\omega, \mathbf{k}) = \delta^{ij} \left(1 - \frac{\omega_p^2}{\omega^2} \right) - \frac{g^2}{2\omega^2} \sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha} \left(\frac{\tilde{v}_\alpha^i k^j + \tilde{v}_\alpha^j k^i}{\omega - \mathbf{k} \cdot \tilde{\mathbf{v}}_\alpha} + \frac{(\omega^2 - \mathbf{k}^2) \tilde{v}_\alpha^i \tilde{v}_\alpha^j}{(\omega - \mathbf{k} \cdot \tilde{\mathbf{v}}_\alpha)^2} \right)$$

Two-stream system

$$\tilde{\mathbf{v}}_+, \tilde{n}_+, \tilde{\epsilon}_+, \tilde{p}_+$$

$$\tilde{\mathbf{v}}_-, \tilde{n}_-, \tilde{\epsilon}_-, \tilde{p}_-$$

$$\tilde{\mathbf{v}} \equiv \tilde{\mathbf{v}}_+ = -\tilde{\mathbf{v}}_-$$

$$\tilde{\mathbf{u}} = \tilde{\gamma} \tilde{\mathbf{v}}$$

$$\tilde{n} \equiv \tilde{n}_+ = \tilde{n}_-, \quad \tilde{\epsilon} \equiv \tilde{\epsilon}_+ = \tilde{\epsilon}_-, \quad \tilde{p} \equiv \tilde{p}_+ = \tilde{p}_-$$

$$\begin{aligned} \epsilon^{ij}(\omega, \mathbf{k}) = & \delta^{ij} \left(1 - \frac{\omega_p^2}{\omega^2} \right) - \frac{g^2}{2\omega^2} \frac{\tilde{n}^2}{\tilde{\epsilon} + \tilde{p}} \left(\frac{\tilde{v}^i k^j + \tilde{v}^j k^i}{\omega - \mathbf{k} \cdot \mathbf{v}} + \frac{(\omega^2 - \mathbf{k}^2) \tilde{v}^i \tilde{v}^j}{(\omega - \mathbf{k} \cdot \mathbf{v})^2} \right. \\ & \left. - \frac{\tilde{v}^i k^j + \tilde{v}^j k^i}{\omega + \mathbf{k} \cdot \mathbf{v}} + \frac{(\omega^2 - \mathbf{k}^2) \tilde{v}^i \tilde{v}^j}{(\omega + \mathbf{k} \cdot \mathbf{v})^2} \right) \end{aligned}$$

$$\mathbf{k} \perp \mathbf{v}$$

$$\tilde{\mathbf{v}} = (0, 0, \tilde{v}), \quad \mathbf{k} = (k, 0, 0)$$

Dispersion equation

$$(\omega^2 - \omega_p^2)(\omega^2 - \omega_p^2 - k^2) \left(\omega^2 - \omega_p^2 - k^2 - \lambda^2 \frac{k^2 - \omega^2}{\omega^2} \right) = 0$$

stable longitudinal
mode

$$\omega^2 = \omega_p^2$$

$$\lambda \equiv \omega_p \tilde{v}$$

stable transverse
modes

$$\omega^2 = \omega_p^2 + k^2$$

$$\omega_+^2 = \frac{1}{2} \left(\omega_p^2 - \lambda^2 + k^2 + \sqrt{(\omega_p^2 - \lambda^2 + k^2)^2 + 4\lambda^2 k^2} \right)$$

unstable transverse,
filamentation
mode

$$\omega_-^2 = \frac{1}{2} \left(\omega_p^2 - \lambda^2 + k^2 - \sqrt{(\omega_p^2 - \lambda^2 + k^2)^2 + 4\lambda^2 k^2} \right)$$

$$\mathbf{k} \parallel \mathbf{v}$$

$$\tilde{\mathbf{v}} = (0, 0, \tilde{v}), \quad \mathbf{k} = (0, 0, k)$$

Dispersion equation

$$(\omega^2 - \omega_p^2 - k^2)^2 \left(1 - \frac{\omega_0^2}{(\omega - k\tilde{v})^2} + \frac{\omega_0^2}{(\omega + k\tilde{v})^2} \right) = 0$$

stable transverse
modes

$$\omega^2 = \omega_p^2 + k^2$$

$$\omega_0^2 \equiv \omega_p^2 (1 - \tilde{v}^2) / 2$$

stable longitudinal
mode

$$\omega_+^2 = \omega_0^2 + k^2 \tilde{v}^2 + \sqrt{\omega_0^2 + 4k^2 \tilde{v}^2}$$

unstable longitudinal
mode

$$\omega_+^2 = \omega_0^2 + k^2 \tilde{v}^2 - \sqrt{\omega_0^2 + 4k^2 \tilde{v}^2}$$

Conclusions

Chromohydrodynamics is much simpler than the kinetic theory.

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Chromohydrodynamics is dynamically nontrivial.

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Chromohydrodynamics provides alternative approach
to study nonequilibrium QGP.